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Generalized triangulations and diagonal-free subsets of stack polyominoes

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Abstract

For $n \geqslant 3$, let Ω_n be the set of line segments between vertices in a convex n-gon. For $j \geqslant 1$, a *j-crossing* is a set of j distinct and mutually intersecting line segments from Ω_n such that all $2j$ endpoints are distinct. For $k \geqslant 1$, let $\Delta_{n,k}$ be the simplicial complex of subsets of Ω_n not containing any $(k+1)$-crossing. For example, $\Delta_{n,1}$ has one maximal set for each triangulation of the n-gon. Dress, Koolen, and Moulton were able to prove that all maximal sets in $\Delta_{n,k}$ have the same number $k(2n-2k-1)$ of line segments. We demonstrate that the number of such maximal sets is counted by a $k \times k$ determinant of Catalan numbers. By the work of Desainte-Catherine and Viennot, this determinant is known to count quite a few other objects including fans of non-crossing Dyck paths. We generalize our result to a larger class of simplicial complexes including some of the complexes appearing in the work of Herzog and Trung on determinantal ideals.

Keywords: Enumeration; Polygon; Triangulation; Associahedron; Catalan number; Hankel determinant

1. Introduction

For $n \geqslant 3$, let $x_0, x_1, \ldots, x_{n-1}$ be points arranged evenly spaced in clockwise direction around a circle; the points constitute the vertex set of a regular n-gon as illustrated in Fig. 1.

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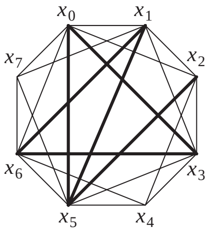

1	2	3	4	5	6	7	
1	1	**1**	-	**1**	1	1	0
	1	1	-	**1**	**1**	1	1
		1	1	**1**	-	-	2
			1	1	**1**	-	3
				1	1	-	4
					1	1	5
						1	6

Fig. 1. A generalized triangulation for $k = 2$ and $n = 8$ and the corresponding subset of the polyomino Ω_8. The nonbold edges are always present in any generalized triangulation, as they cannot be part of a 3-crossing.

For two points x and y, the set $(x, y) = \{(1-t)x + ty : 0 < t < 1\}$ is the *(open) line segment* between x and y. Two line segments (x_a, x_b) and (x_c, x_d) with $0 \leqslant a < b \leqslant n-1$ and $0 \leqslant c < d \leqslant n-1$ *cross* if $a < c < b < d$ or $c < a < d < b$; note that this has a natural geometric interpretation. For any positive integer j, a *j-crossing* is a set consisting of j line segments such that any two segments in the set cross. For example, $\{(x_0, x_5), (x_1, x_6), (x_2, x_7)\}$ is a 3-crossing, whereas $\{(x_0, x_5), (x_1, x_7), (x_2, x_7)\}$ is not, as (x_1, x_7) and (x_2, x_7) do not cross.

Let k be a positive integer. The purpose of this paper is to enumerate maximal sets of line segments between vertices in the n-gon such that no subset forms a $(k+1)$-crossing. For $k = 1$, such sets are just ordinary triangulations of the n-gon. For this reason, we will occasionally refer to these sets as *generalized* triangulations. We give an example for $k = 2$ and $n = 8$ in Fig. 1.

Generalized triangulations appear in the work of Capoyleas and Pach [3] and Dress et al. [9–11]; see the first paper [3] for some further history and background. The listed papers contain several intriguing results about the objects, the most remarkable result being that all generalized triangulations have the same size for any given fixed parameters n and k [10].

In the present paper, we show that the objects under consideration are counted by certain fans of non-crossing lattice paths. Desainte-Catherine and Viennot [6] enumerated these fans using the lattice path determinant formula of Lindström [17] and Gessel-Viennot [12]; see also the work of Ghorpade and Krattenthaler [13]. As a consequence, the number of generalized triangulations for any fixed n and k is equal to a Hankel determinant of Catalan numbers; see Corollary 16 in Section 4.2. This result is a special case of a more general result about "$(k+1)$-diagonal-free subsets of stack polyominoes", which we will now describe in greater detail.

From sets of line segments to subsets of polyominoes: *A polyomino* [2] is a finite subset of $\mathbb{Z}^2$. We want to translate our problem to an equivalent problem defined in terms of subsets of a given polyomino. For later convenience, we adopt the matrix convention for indexing rows and columns in $\mathbb{Z}^2$; row a is just below row $a-1$, column b is just to the right of column $b-1$, and ab refers to the element in row a and column b. The translation is as follows: For vertices x_a and x_b such that $a < b$, we identify the line segment (x_a, x_b) with

[2] For generality, we do not restrict this definition to "connected" subsets of $\mathbb{Z}^2$, as is normally the case.

	1	2	3
0	1		
1		1	
2	$*$		1

	1	2	3
0	1		
1		1	
2	$*$		1

Fig. 2. {01, 12, 23} is a 3-diagonal in the polyomino to the left but not in the polyomino to the right, as the latter polyomino does not contain the position 21 marked with "$*$".

the lattice point $ab = (a, b)$. More generally, we identify a set of line segments in the n-gon with a subset of the polyomino

$$\Omega_n = \{ij : i \in [0, n-2],\ j \in [i+1, n-1]\}, \tag{1}$$

$[a, b]$ denotes the set $\{c \in \mathbb{Z} : a \leqslant c \leqslant b\}$. See Fig. 1 for an example; we illustrate each element in the polyomino as a square and assign a given square the value "1" whenever the corresponding lattice point is part of the given subset and the value "-" otherwise.

From $(k+1)$*-crossings to* $(k+1)$*-diagonals*: A crucial observation about crossings is the following: For $a < b$, $c < d$, and $a \leqslant c$, two line segments (x_a, x_b) and (x_c, x_d) cross if and only if $a < c$, $b < d$, and the 2×2 square $\{ab, ad, cb, cd\} = \{a, c\} \times \{b, d\}$ is a subset of Ω_n. Namely, cb being contained in Ω_n means exactly that $c < b$. This observation gives a hint as how to generalize to an arbitrary polyomino Λ.

Definition 1. Two elements ab and cd form a 2-*diagonal* in the polyomino Λ if $a < c$, $b < d$, and the 2×2 square $\{ab, ad, cb, cd\} = \{a, c\} \times \{b, d\}$ is a subset of Λ. More generally, $a_1b_1, \ldots, a_rb_r$ form an r-*diagonal* in Λ if the following hold:

- $a_1 < a_2 < \cdots < a_r$ and $b_1 < b_2 < \cdots < b_r$
- The $r \times r$ square $\{a_ib_j : i, j \in [1, r]\} = \{a_1, \ldots, a_r\} \times \{b_1, \ldots, b_r\}$ is a subset of Λ.

See Fig. 2 for an illustration.

For $k \geqslant 1$, define $\Delta_{\Lambda,k}$ as the family of subsets σ of Λ such that σ does not contain any set forming a $(k+1)$-diagonal in Λ. Clearly, $\Delta_{\Lambda,k}$ is a simplicial complex. Let $\Delta_{n,k}$ denote the simplicial complex $\Delta_{\Omega_n,k}$, where Ω_n is defined in (1); this is exactly the complex of full and partial generalized triangulations of the n-gon. Another important example is $\Delta_{A_n,k}$, where $A_n = \{ij : i \in [0, n-2],\ j \in [1, n-1-i]\}$. See Fig. 3 for a few simple examples with $k = 1$. Let $\mathcal{M}_{\Lambda,k}$ denote the family of maximal faces in $\Delta_{\Lambda,k}$; $\mathcal{M}_{n,k}$ denotes the family of maximal faces in $\Delta_{n,k}$.

Remark. The restriction in Definition 1 to squares completely contained in Λ may remind the reader of a similar restriction in the work of Herzog and Trung on determinantal ideals on "ladders" [14, Section 4]. Specifically, our complex $\Delta_{\Lambda,k}$ coincides with their complex $\Delta_M(Y)$; $Y = \Lambda$ and $M = [1, \ldots, k|1, \ldots, k]$. We also mention the work of Backelin, West, and Xin on pattern-avoiding permutations in a Young tableaux [1]. We do not know whether there is a deeper connection between their work and the present paper.

Stack and moon polyominoes: From now on, we concentrate on two special kinds of polyominoes that we refer to as *stack* and *moon* polyominoes, respectively. For a polyomino Λ and a column index j, write $\Lambda^j = \{i : ij \in \Lambda\}$; this is the set of row indices i such that

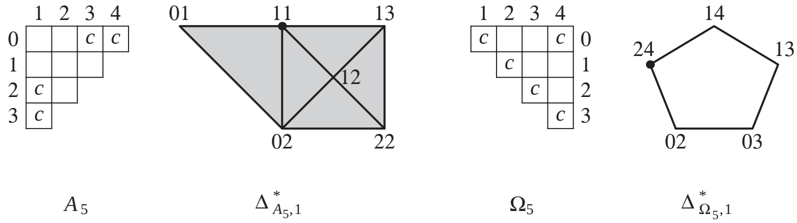

Fig. 3. The simplicial complexes $\Delta^*_{A_5,1}$ and $\Delta^*_{\Omega_5,1}$ that we obtain from $\Delta_{A_5,1}$ and $\Delta_{5,1} = \Delta_{\Omega_5,1}$ by removing all cone points (marked with "c"). Note that $\Delta_{A_5,1}$ and $\Delta_{5,1}$ both have five maximal faces.

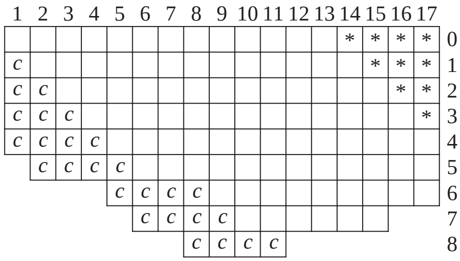

Fig. 4. The stack polyomino $\Lambda = (5, 6, 6, 6, 7, 8, 8, 9, 9, 9, 9, 8, 8, 8, 8, 7, 7)$; the column support of Λ is $[1, 17]$. The elements in the corner $\Gamma_{\Lambda,4}$ (see Section 1.1) are marked with "$*$". Other cone points in $\Delta_{\Lambda,4}$ are marked with c.

$ij \in \Lambda$. Analogously, write $\Lambda_i = \{j : ij \in \Lambda\}$; this is the set of column indices j such that $ij \in \Lambda$.

Definition 2. A polyomino Λ is *column-convex* if each column Λ^j is empty or an interval $[a, b]$ for some $a \leqslant b$. Λ is *row-convex* if the corresponding property holds for each row Λ_i. If Λ is both column-convex and row-convex, then Λ is *convex*. Λ is *intersection-free* if, for any two column indices j and j', either $\Lambda^j \subseteq \Lambda^{j'}$ or $\Lambda^j \supseteq \Lambda^{j'}$. Equivalently, Λ is intersection-free if the analogous property holds for every two rows Λ_i and $\Lambda_{i'}$.

Definition 3. A polyomino Λ is a *moon polyomino* if Λ is intersection-free and convex. If in addition each Λ^j is of the form $[0, \lambda_j - 1]$ for some non-negative integer λ_j, then Λ is a *stack polyomino*.

See Fig. 4 for a stack polyomino; proceed to Figs. 7 and 6 for some moon polyominoes.

For a polyomino Λ, refer to the set $\{j : \Lambda^j \neq \emptyset\}$ as the *column support* of Λ and to the set $\{i : \Lambda_i \neq \emptyset\}$ as the *row support*. For any moon polyomino Λ, we may assume that the column support is $[1, n]$ for some n; the structure of $\Delta_{\Lambda,k}$ is preserved under translation of Λ. By convexity, a polyomino

$$\Lambda = \{ij : j \in [1, n], i \in [0, \lambda_j - 1]\} \tag{2}$$

with column support $[1, n]$ is a stack polyomino if and only if the sequence $(\lambda_1, \ldots, \lambda_n)$ is unimodal; $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \geqslant \lambda_{j+1} \geqslant \cdots \geqslant \lambda_n \geqslant 1$ for some $j \in [1, n]$. We identify a stack polyomino Λ with the corresponding sequence $(\lambda_1, \ldots, \lambda_n)$ of integers. For example, $\Omega_n = (1, 2, \ldots, n-1)$, where Ω_n is defined in (1). For another example, see Fig. 4.

Definition 4. We obtain the *content* of a stack polyomino Λ by arranging the elements $\lambda_1, \ldots, \lambda_n$ in decreasing order. To define the content of a moon polyomino Λ, let λ_j be the number of elements in the jth column of Λ. By the properties of moon polyominoes, the sequence $(\lambda_1, \ldots, \lambda_n)$ defines a stack polyomino Λ'; we define the content of Λ as the content of Λ'.

For example, the content of $(2, 3, 6, 5, 5, 3, 3)$ is $(6, 5, 5, 3, 3, 3, 2)$.

Summary of results: In Section 3, we derive the crucial property that $\Delta_{\Lambda,k}$ is a pure simplicial complex whenever Λ is a stack polyomino and $k \geqslant 1$; this generalizes the corresponding result by Dress et al. [10] about $\Delta_{n,k}$.

In Section 4, we present our main result: Any two stack polyominoes Λ and Λ' with the same content satisfy $|\mathcal{M}_{\Lambda,k}| = |\mathcal{M}_{\Lambda',k}|$; this is Theorem 14. This result implies that $|\mathcal{M}_{n,k}|$ is equal to a certain $k \times k$ determinant of Catalan numbers as given in Corollary 16. Namely, this is known to be true for $|\mathcal{M}_{A_n,k}|$, where $A_n = (n-1, n-2, \ldots, 2, 1)$; whenever $\Lambda = (\lambda_1, \ldots, \lambda_n)$ is a weakly decreasing sequence, one may compute $|\mathcal{M}_{\Lambda,k}|$ via the lattice path determinant formula due to Lindström [17] and Gessel-Viennot [12]. We have some hope that our results remain true for general moon polyominoes, but this problem is still open.

In Section 6, we give a refined result about the number of maximal faces with a certain number of elements in each row. Specifically, the number of faces in $\mathcal{M}_{\Lambda,k}$ with δ_i elements in row i for each i coincides with the corresponding number of faces in $\mathcal{M}_{\Lambda',k}$ whenever Λ and Λ' are stack polyominoes with the same content.

1.1. Notation and basic concepts

A *simplicial complex* on a finite set V is a nonempty family of subsets of V closed under deletion of elements. Members of a simplicial complex Σ are called *faces*. The *dimension* of a face σ is defined as $|\sigma| - 1$. The dimension of a complex Σ is the maximal dimension of any face in Σ. A complex is *pure* if all maximal faces have the same dimension. For a face $\sigma \in \Sigma$, the *link* $\mathrm{lk}_\Sigma(\sigma)$ is the set of all τ in Σ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in \Sigma$. An element x in V is a *cone point* in Σ if $\sigma \cup \{x\}$ belongs to Σ whenever σ belongs to Σ.

For integers a, b, recall that $[a, b]$ denotes the interval $\{c \in \mathbb{Z} : a \leqslant c \leqslant b\}$. Define (a, b) as the open interval $\{c \in \mathbb{Z} : a < c < b\}$ and $(a, b]$ and $[a, b)$ as the half-open intervals $\{c \in \mathbb{Z} : a < c \leqslant b\}$ and $\{c \in \mathbb{Z} : a \leqslant c < b\}$, respectively.

For a row set I, we define an *I-diagonal* to be an $|I|$-diagonal E of the form $\{iy_i : i \in I\}$; for each $i \in I$, there is exactly one element from row i in E.

For a stack polyomino $\Lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_j \geqslant k$ for all j, define the *corner* $\Gamma_{\Lambda,k}$ of Λ as the set $\{(i-1)j : i \in [1, k], j \in [n-k+i, n]\}$; see Fig. 4 for an example. Clearly, all elements in the corner of Λ are cone points in $\Delta_{\Lambda,k}$. For example, no 5-diagonal can contain an element in the corner in Fig. 4. However, as the figure illustrates, there are typically many other cone points. We let $\mathsf{Cone}_{\Lambda,k}$ denote the set of cone points in Λ.

1.2. Related work

For the special case $k = 1$ and $\Lambda = (n-1, n-2, \ldots, 1)$, the objects under consideration have a long history. Indeed, already 250 years ago, Euler proposed the problem of counting triangulations of the n-gon. The problem is for this reason sometimes referred to as "Euler's Polygon Division Problem." The correct answer to the problem is well-known to be C_{n-2}, where $C_m = \binom{2m}{m}/(m+1)$ is the mth Catalan number. There are literally dozens of different proofs of this result; for a few of the more well-known, see [18, Ex. 1.38-40]. As it turns out, there is a nearly inexhaustible amount of combinatorial objects counted by Catalan numbers; Stanley [20,21] provides an extensive list of such objects. Just to give a few examples, we may mention binary trees, Dyck paths, and ρ-avoiding permutations, where $\rho \in S_3$. We refer the reader to Stanton and White [22, Section 3.1] for more information about binary trees. Krattenthaler [15] (and, independently, E. Deutsch) recently discovered elegant bijections between Dyck paths and ρ-avoiding permutations.

For $k \geqslant 1$, we will show that $|\mathcal{M}_{n,k}|$ is counted by a certain determinant of Catalan numbers; see Corollary 16. This determinant counts several known objects, some of the most important objects being perfect matchings in honeycomb graphs, non-crossing fans of Dyck paths, "subdiagonal clouds of points", and certain Young tableaux with bounded height. See [4,5] for details about honeycomb graphs and Desainte-Catherine and Viennot [6,7] for details abut the other three objects. Rolbetzki [19] gives a nice overview and also computes $|\mathcal{M}_{n,k}|$ for $n \leqslant 2k+3$. As far as we know, this is the only previously known non-trivial and general enumerative result about $\mathcal{M}_{n,k}$ for $k \geqslant 2$. We stress that Rolbetzki's proof is *direct*, whereas our proof goes via induction on the size of the polyomino. An important question is whether there exists a more direct proof in the general case.

One may also examine the simplicial complex $\Delta_{n,k}$ from a topological point of view. By a recent result [8], $\Delta_{n,k}$ is a kn-fold cone over a shellable piece-wise linear sphere $\Delta^*_{n,k}$ of dimension $k(n-2k-1)-1$. This extends the result of Dress et al. [10] that $\Delta_{n,k}$ is pure, which in turn extends the result of Capoyleas and Pach [3] that $\Delta_{n,k}$ is of dimension $k(2n-2k-1)-1$. $\Delta^*_{n,k}$ is conjectured to be polytopal (i.e., the boundary complex of a convex polytope) [8]. Since $\Delta^*_{n,1}$ is the boundary complex of the nth associahedron, this is true for $k = 1$. We refer to Lee [16] and Ziegler [24] for explicit polytopal realizations of $\Delta^*_{n,1}$.

2. Auxiliary lemmas

We will frequently use the following fact throughout the paper.

Lemma 5. *Let Λ be a convex polyomino and let $r \geqslant 2$. Let $E = \{a_i b_i : i \in [1,r]\}$ be an r-diagonal in Λ with $a_i < a_{i+1}$ and $b_i < b_{i+1}$ and let $a_0 \leqslant a_1$ and $b_0 \leqslant b_1$ be such that $a_0 b_0, a_0 b_r, a_r b_0 \in \Lambda$. If $E' = \{c_i d_i : i \in [1,r]\}$ satisfies $a_0 \leqslant c_1 < c_2 < \cdots < c_r \leqslant a_r$ and $b_0 \leqslant d_1 < d_2 < \cdots < d_r \leqslant b_r$, then E' is an r-diagonal in Λ.*

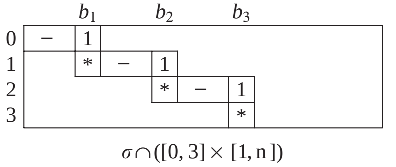

Fig. 5. The situation in Lemma 6 with $p = 3 \leqslant k$. We may add elements marked with a star to $\sigma \in \Delta_{\Lambda,k}$ without introducing any $(k + 1)$-diagonal.

Proof. We need only demonstrate that $c_i d_j \in \Lambda$ for all $i, j \in [1, r]$. Now, by column convexity, the interval $[a_0, a_r]$ is part of Λ^{b_0} and Λ^{b_r}. By row convexity, we obtain from this that $[a_0, a_r] \times [b_0, b_r] \subseteq \Lambda$, which immediately implies that $c_i d_j \in \Lambda$ for all i, j. $\square$

When applying Lemma 5, we use $a_0 = a_1$ and $b_0 = b_1$ unless otherwise stated. In the following lemma, recall that $\mathsf{Cone}_{\Lambda,k}$ is the set of cone points in $\Delta_{\Lambda,k}$.

Lemma 6. *Let $1 \leqslant p \leqslant k$ and $n \geqslant p$. Let Λ be a moon polyomino with column support $[1, n]$ containing the rectangle $[0, p] \times [1, n]$. Let $b_1, b_2, \ldots, b_p$ be integers such that $1 \leqslant b_1 < b_2 < \cdots < b_p \leqslant n$. Suppose $\sigma \in \Delta_{\Lambda,k}$ has the property that $(i - 1)b_i \in \sigma$ for $i \in [1, p]$. If $(\sigma \setminus \mathsf{Cone}_{\Lambda,k}) \cap (\{i\} \times (b_i, b_{i+1})) = \emptyset$ for $i \in [0, p - 1]$ $(b_0 = 0)$, then we may add pb_p to σ without introducing a $(k + 1)$-diagonal.*

Remark. Fig. 5 illustrates the situation.

Proof. By induction on p, we may assume that $ib_i \in \sigma$ for $i \in [1, p - 1]$; the base case is that we assume nothing for $p = 1$. Suppose that pb_p introduces a $(k + 1)$-diagonal, and let E be a $(k + 1)$-diagonal in $\sigma + pb_p$ with as many elements from $B_1 = \{ib_i : i \in [1, p]\}$ as possible. Let a be the smallest row index such that $ay \in E$ for some y. Define $r = \max(\{0\} \cup \{i : ib_i \notin E, i \in [1, p - 1]\})$; it is clear that $r \in [0, p - 1]$.

If there is no element in E of the form ry, then we may replace $\{ib_i : i \in [r + 1, p]\}$ with $\{(i - 1)b_i : i \in [r + 1, p]\}$ and obtain a $(k + 1)$-diagonal in σ, a contradiction; apply Lemma 5 with $a_0 = \min\{r, a\}$.

Suppose $ry \in E$ for some y. If $r = 0$, we have that $1b_1 \in E$, which implies that $y < b_1$. However, by assumption, this means that $0y$ is a cone point and hence not part of any $(k+1)$-diagonal, which yields a contradiction. Thus we must have $r > 0$. Since $(r + 1)b_{r+1} \in E$, we have that $y < b_{r+1}$. Since $\sigma \cap (\{r\} \times (b_r, b_{r+1}))$ only contains cone points, we have that $y < b_r$; $y \neq b_r$ by assumption. This means that we can replace ry with rb_r and still have a $(k + 1)$-diagonal; use Lemma 5. This is a contradiction to the maximality of $E \cap B_1$, and we are done. $\square$

Lemma 7. *Let $1 \leqslant k \leqslant n$. Let Λ be a moon polyomino with column support $[1, n]$ containing the rectangle $[0, k] \times [1, n]$. For any face $\sigma \in \Delta_{\Lambda,k}$, there is at most one set*

$$B = \bigcup_{i=1}^{k} \{(i - 1)b_i, ib_i\}$$

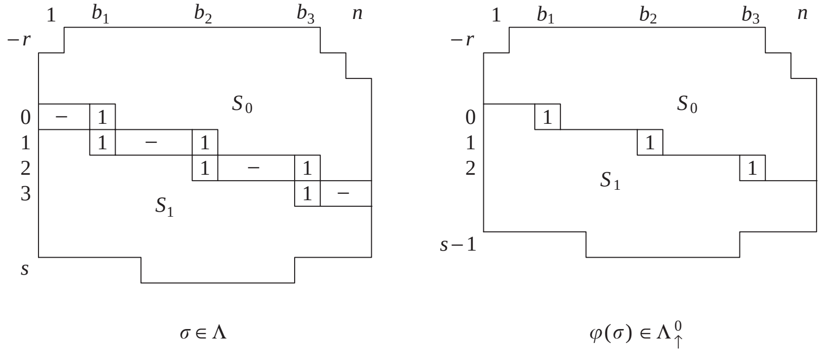

Fig. 6. The function φ in the proof of Lemma 9 for $k = 3$; S_0 remains fixed, whereas S_1 is moved one step up.

satisfying $1 \leqslant b_1 < b_2 < \cdots < b_k \leqslant n$ *such that* $B \subset \sigma$. *Also,* σ *contains no elements from* $\{i\} \times (b_i, b_{i+1})$ *for* $i \in [0, k]$; $b_0 = 0$ *and* $b_{k+1} = n + 1$.

Remark. The polyomino to the left in Fig. 6 illustrates the situation.

Proof. Suppose $B \subset \sigma$. Then, for each $i \in [0, k]$, $\{(r-1)b_r : r \in [1, i]\} \cup \{rb_r : r \in [i+1, k]\}$ is a k-diagonal. In particular, the last statement in the lemma follows; σ contains no element from the set $\{i\} \times (b_i, b_{i+1})$, as such an element would create a $(k+1)$-diagonal. As a consequence, the set B is uniquely defined. Namely, we have that

$$b_i = \min\{b : (i-1)b \in \sigma, b > b_{i-1}\} \tag{3}$$

for $i \in [1, k]$. $\square$

In the following lemma, we restrict our attention to stack polyominoes.

Lemma 8. *Let* $1 \leqslant k \leqslant n$. *Let* Λ *be a stack polyomino with column support* $[1, n]$ *containing the rectangle* $[0, k] \times [1, n]$. *For any maximal face* $\sigma \in \mathcal{M}_{\Lambda,k}$, *there is a unique set B with properties as in Lemma* 7.

Proof. For uniqueness, apply Lemma 7. For existence, note that (3) gives a well-defined sequence $(b_1, \ldots, b_n)$. Namely, since $\Gamma_{\Lambda,k} \subseteq \sigma$, we easily deduce that $i \leqslant b_i \leqslant n - k + i$. It remains to show that $ib_i \in \sigma$. However, this is an immediate consequence of Lemma 6, and we are done. $\square$

We conjecture that Lemma 8 holds also for moon polyominoes. The proof is likely to be substantially harder for this case, as there is no corner to rely on.

3. $\Delta_{\Lambda,k}$ Is pure

The purpose of this section is to show that $\Delta_{\Lambda,k}$ is a pure complex whenever Λ is a stack polyomino; this generalizes the corresponding result about $\Delta_{n,k}$ by Dress et al. [10]. Another special case is that the columns in Λ are arranged in decreasing order. In this case, Λ has the structure of a planar distributive lattice; the meet operation is given by $ab \wedge cd = (\max\{a, c\})(\min\{b, d\})$. The j-diagonals are exactly the antichains of size j, which yields that $\Delta_{\Lambda,k}$ is the complex of subsets of Λ without any antichain of size $k + 1$. By a result of Björner [2, Theorem 7.1], $\Delta_{\Lambda,k}$ is pure and shellable.

Before being able to establish purity, we need to introduce some notation. For a polyomino Λ and a row index i, let $\Lambda^i_\uparrow$ denote the polyomino that we obtain by removing row i from Λ and moving everything below row i one step up; row $i + 1$ is moved to row i, row $i + 2$ is moved to row $i + 1$, and so on. For a column index j, let $\Lambda^{j\leftarrow}$ denote the polyomino that we obtain by removing column j from Λ and by moving everything to the right of column j one step to the left. Fig. 7 provides a simple example.

The following lemma is crucial for our proof that $\Delta_{\Lambda,k}$ is pure. Note that we consider general moon polyominoes in this lemma, not only stack polyominoes.

Lemma 9. *Let $1 \leqslant k \leqslant n$. Let Λ be a moon polyomino with column support $[1, n]$ containing the rectangle $[0, k] \times [1, n]$. Let $b_1, \ldots, b_k$ be integers such that $1 \leqslant b_1 < b_2 < \cdots < b_k \leqslant n$. Let $B_0 = \{(i - 1)b_i : i \in [1, k]\}$ and $B_1 = \{ib_i : i \in [1, k]\}$. Then there exists an isomorphism $\varphi : \mathrm{lk}_{\Delta_{\Lambda,k}}(B_0 \cup B_1) \to \mathrm{lk}_{\Delta_{\Lambda',k}}(B_0)$, where $\Lambda' = \Lambda^0_\uparrow$.*

Proof. Write $\Sigma_1 = \mathrm{lk}_{\Delta_{\Lambda,k}}(B_0 \cup B_1)$ and $\Sigma_0 = \mathrm{lk}_{\Delta_{\Lambda',k}}(B_0)$. Let D be the union of B_0, B_1, and the sets $\{i\} \times (b_i, b_{i+1})$ for $i \in [0, k]$; $b_0 = 0$ and $b_{k+1} = n + 1$. By Lemma 7, no elements from D are present in Σ_1. The elements in D form a path from the upper left corner $(0, 1)$ down to the lower right corner (k, n) in the rectangle $[0, k] \times [1, n]$; see Fig. 6. Let S_0 be the region in Λ to the right and above D and let S_1 be the region in Λ to the left and below D;

$$S_0 = \bigcup_{i<0} (\{i\} \times \Lambda_i) \cup \bigcup_{i=0}^{k-1} \{ij : j \in \Lambda_i,\ j > b_{i+1}\},$$

$$S_1 = \bigcup_{i>k} (\{i\} \times \Lambda_i) \cup \bigcup_{i=1}^{k} \{ij : j \in \Lambda_i,\ j < b_i\}.$$

Note that all vertices in Σ_1 lie in $S_0 \cup S_1 = \Lambda \setminus D$. We obtain our isomorphism by moving $B_1 \cup S_1$ one step up. Specifically, let φ be defined on elements in $B_0 \cup B_1 \cup S_0 \cup S_1$ by

$$\varphi(ij) = \begin{cases} ij & \text{if } ij \in B_0 \cup S_0, \\ (i-1)j & \text{if } ij \in B_1 \cup S_1. \end{cases} \tag{4}$$

This gives a bijection from $B_0 \cup S_0 \cup S_1 = \Lambda \setminus (D \setminus B_0)$ to $\Lambda^0_\uparrow$ and a 2-to-1 surjection from $B_0 \cup B_1$ to B_0.

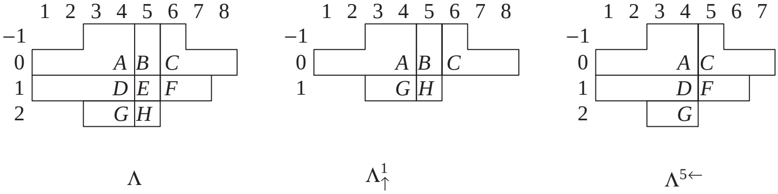

Fig. 7. We obtain $\Lambda^1_\uparrow$ from Λ by removing row 1 and moving everything below this row (i.e., row 2) one step up. Analogously, to obtain $\Lambda^{5\leftarrow}$ we remove column 5 and move the columns to the right of this column one step to the left.

We want to prove that φ induces an isomorphism $\Sigma_1 \to \Sigma_0$. First, we show that $\varphi(\Sigma_1) \subseteq \Sigma_0$, meaning that φ does not introduce any $(k+1)$-diagonals. This is the least intuitive part of the construction, because φ may introduce quite a few smaller diagonals.

Let σ be a face in Σ_1 and assume to the contrary that $\varphi(\sigma) \cup B_0$ does contain a $(k+1)$-diagonal E. Choose E such that $E \cap B_0$ is maximal. Since $\sigma \cup B_0$ does not contain any $(k+1)$-diagonal, there must be two elements $ix, iy \in \sigma \cup B_0$ with $x < y$ and $i \in [1, k-1]$ such that $\varphi(ix) = (i-1)x \in E$ and $\varphi(iy) = iy \in E$. Note that $x \leqslant b_i$ and $y \geqslant b_{i+1}$. Since $ib_i \in B_1$ and $ix \notin B_1$, we have $x < b_i$. Now, replace $(i-1)x$ with $(i-1)b_i$ in E; this yields a new $(k+1)$-diagonal, because $b_i \in (x, y)$ (apply Lemma 5). Yet, $(i-1)b_i \in B_0 \setminus E$, which is a contradiction to the maximality of $E \cap B_0$.

It remains to show that $\Sigma_0 \subseteq \varphi(\Sigma_1)$, meaning that φ maps $(k+1)$-diagonals to $(k+1)$-diagonals. For this, it suffices to prove that φ maps 2-diagonals to 2-diagonals. Suppose that xy and zw form a 2-diagonal in Λ; $x < z$ and $y < w$. The only way for $\varphi(xy)$ and $\varphi(zw)$ not to form a 2-diagonal in $\Lambda^0_\uparrow$ would be that the two elements appear in the same row. This is possible only if $z = x+1$, $\varphi(xy) = xy$, and $\varphi((x+1)w) = xw$. However, then $y \geqslant b_{x+1}$ and $w \leqslant b_{x+1}$ by definition, which implies that $y \geqslant w$, clearly a contradiction. □

Theorem 10. *Let $k \geqslant 1$ and let Λ be a stack polyomino with column support $[1, n]$ and content $\mu = (\mu_1, \ldots, \mu_n)$. Then $\Delta_{\Lambda,k}$ is pure of dimension*

$$d_{\mu,k} = \sum_{i=1}^{k} \mu_i + \sum_{i=k+1}^{n} \min\{\mu_i, k\} - 1.$$

Proof. The theorem is obvious if $n \leqslant k$, because then all elements are cone points in $\Delta_{\Lambda,k}$. Assume that $n \geqslant k+1$ and write $\Lambda = (\lambda_1, \ldots, \lambda_n)$.

First, suppose that $\lambda_1 \leqslant k$ or $\lambda_n \leqslant k$; assume the latter and assume that $\lambda_n = \mu_n$; the case $\lambda_1 = \mu_n \leqslant k$ is treated analogously. Then all elements in $[1, \lambda_n] \times \{n\}$ are cone points in $\Delta_{\Lambda,k}$. In particular, $\mathrm{lk}_{\Delta_{\Lambda,k}}([1, \lambda_n] \times \{n\}) = \Delta_{\Lambda',k}$, where $\Lambda' = (\lambda_1, \ldots, \lambda_{n-1})$. The content of Λ' is $\mu' = (\mu_1, \ldots, \mu_{n-1})$; thus by induction, $\Delta_{\Lambda',k}$ is pure of dimension $d_{\mu',k} = d_{\mu,k} - \mu_n = d_{\mu,k} - \lambda_n$, which implies that $\Delta_{\Lambda,k}$ is pure of dimension $d_{\mu,k}$ as desired.

Next, suppose that $\lambda_j \geqslant k+1$ for all j. Let σ be a maximal face in $\Delta_{\Lambda,k}$ and let $B = B_0 \cup B_1$ be as in Lemma 8; B_0 and B_1 are defined as in Lemma 9. We want to prove that

σ has dimension $d_{\mu,k}$. For this, it suffices to prove that the link $\mathrm{lk}_{\Delta_{\Lambda,k}}(B_0 \cup B_1)$ is pure of dimension $d_{\mu,k} - |B_0 \cup B_1| = d_{\mu,k} - 2k$. By Lemma 9, $\mathrm{lk}_{\Delta_{\Lambda,k}}(B_0 \cup B_1)$ is isomorphic to $\mathrm{lk}_{\Delta_{\Lambda',k}}(B_0)$, where $\Lambda' = \Lambda^0_\uparrow$. By induction, $\Delta_{\Lambda',k}$ is pure of dimension

$$\sum_{i=1}^{k}(\mu_i - 1) + \sum_{i=k+1}^{n} \min\{\mu_i - 1, k\} - 1 = \sum_{i=1}^{k} \mu_i - k + \sum_{i=k+1}^{n} k - 1 = d_{\mu,k} - k.$$

As a consequence, $\mathrm{lk}_{\Delta_{\Lambda',k}}(B_0)$ is pure of dimension $d_{\mu,k} - k - |B_0| = d_{\mu,k} - 2k$, and we are done. $\square$

4. Main results

Recall that $\mathcal{M}_{\Lambda,k}$ is the family of maximal faces in $\Delta_{\Lambda,k}$. The main achievement of this paper is the following result, which is Theorem 14:

- *Let $k \geqslant 1$ and let Λ and Λ' be stack polyominoes with the same content. Then* $|\mathcal{M}_{\Lambda,k}| = |\mathcal{M}_{\Lambda',k}|$.

In Section 4.1, we present a refinement of Theorem 14; in Section 4.2, we show that this refinement implies the theorem.

4.1. The refined theorem

Let $k \geqslant 1$. Throughout this section, $\Lambda = (\lambda_1, \dots, \lambda_n)$ is a stack polyomino with column support $[1, n]$ such that $n \geqslant k$. We will divide the family $\mathcal{M}_{\Lambda,k}$ into smaller families according to the local structure of a maximal face σ within the rectangle $[0, k] \times [1, n]$. Specifically, we will introduce two vectors $\mathsf{s}(\sigma)$ and $\mathsf{t}(\sigma)$ of integers that count certain diagonals in σ.

A *k-vector* $\mathsf{u} = (u_1, \dots, u_k)$ is a vector consisting of k integers. u is *weakly decreasing* if $u_i \geqslant u_{i+1}$ for $i \in [1, k-1]$. Let $\mathcal{WD}_k$ denote the set of weakly decreasing k-vectors such that all coefficients are non-negative. For k-vectors $\mathsf{u} = (u_1, \dots, u_k)$ and $\mathsf{v} = (v_1, \dots, v_k)$, say that $\mathsf{u} \geqslant \mathsf{v}$ if $u_i \geqslant v_i$ for $i \in [1, k]$. $\mathsf{u} > \mathsf{v}$ means that $\mathsf{u} \geqslant \mathsf{v}$ with strict inequality $u_i > v_i$ for at least one index i.

The vector $\mathsf{s}(\sigma)$: Assume that $\lambda_j \geqslant k$ for all $j \in [1, n]$. For $\sigma \in \mathcal{M}_{\Lambda,k}$, we define a k-vector $\mathsf{s} = \mathsf{s}(\sigma) \in \mathcal{WD}_k$ such that the value s_r is the maximal number of mutually disjoint $[0, r-1]$-diagonals in $\sigma \setminus \Gamma_{\Lambda,k}$ for each $r \in [1, k]$. We form the diagonals as follows. First, we list all elements in row 0 in $\sigma \setminus \Gamma_{\Lambda,k}$ in increasing order. Next, we take the longest and leftmost sequence of elements in row 1 with the property that the jth element in the sequence forms a diagonal with the jth element from row 0. Continue in the same manner for all $i \in [1, k]$, choosing the longest and leftmost sequence of elements in row $i-1$ such that the jth element forms a diagonal with the previously chosen jth elements from row $0, \dots, i-2$.

Formally, let $b_{i,0} = b_{i,0}(\sigma) = 0$ for $i \in [1, k]$ and $b_{0,j} = 0$ for all j. Proceed recursively as follows. If $b_{i,j-1}$ and $b_{i-1,j}$ have been defined and $b_{i,j-1} < n - k + i$ (equivalently,

$(i-1)b_{i,j-1} \notin \Gamma_{\Lambda,k}$), let

$$b_{i,j} = b_{i,j}(\sigma) = \min\{b : (i-1)b \in \sigma, b > \max\{b_{i-1,j}, b_{i,j-1}\}\}. \tag{5}$$

Note that $b_{i,1} = b_i$, where $b_i = b_i(\sigma)$ is defined as in (3). If $b_{i,j-1} = n-k+i$ or if either of $b_{i,j-1}$ or $b_{i-1,j}$ is undefined, then we leave $b_{i,j}$ undefined. For each $i \in [1,k]$, let $s_i = s_i(\sigma)$ be the largest index such that b_{i,s_i+1} is defined; we define $\mathsf{s}(\sigma) = (s_1, \ldots, s_k)$. It is obvious that $\mathsf{s}(\sigma) \in \mathcal{WD}_k$ for any $\sigma \in \mathcal{M}_{\Lambda,k}$.

Lemma 11. *For each $\sigma \in \mathcal{M}_{\Lambda,k}$ and each $i \in [1,k]$, we have that $b_{i,s_i+1} = n-k+i$.*

Proof. The lemma is clearly true for $i = 1$, as $b_{1,j}$ is simply the jth element in row 0 from σ. For $i > 1$, we have by induction that $b_{i-1,s_{i-1}+1} = n-k+i-1$. Since each element $b_{i-1,j}$ is at most $n-k+i-1$, it is clear that whenever $b_{i,j-1}$ and $b_{i-1,j}$ are defined and $b_{i,j-1} < n-k+i$, we must have that $b_{i,j} \leqslant n-k+i$. As a consequence, the very last value $b_{i,j}$ defined in row i must be equal to $n-k+i$; if $b_{i,s_{i-1}}$ has been defined and is smaller than $n-k+i$, then we certainly have that $b_{i,s_{i-1}+1} = n-k+i$, because $b_{i-1,s_{i-1}+1} = n-k+i-1$. □

The vector $\mathsf{t}(\sigma)$: When $\lambda_j \geqslant k+1$ for all $j \in [1,n]$, we also define a vector $\mathsf{t}(\sigma) \in \mathcal{WD}_k$. This vector is defined in exactly the same manner as $\mathsf{s}(\sigma)$, except that we consider $[1,r]$-diagonals with all elements *ij* satisfying $j < b_i = b_{i,1}$.

Formally, define $a_{i,0} = a_{i,0}(\sigma) = 0$ for $i \in [1,r]$ and $a_{0,j} = 0$ for all j. Proceed recursively as follows. If $a_{i,j-1}$ and $a_{i-1,j}$ have been defined and $a_{i,j-1} < b_{i,1} = b_i$, let

$$a_{i,j} = a_{i,j}(\sigma) = \min\{a : ia \in \sigma, a > \max\{a_{i-1,j}, a_{i,j-1}\}\}. \tag{6}$$

If $a_{i,j-1} = b_{i,1}$ or if either of $a_{i,j-1}$ or $a_{i-1,j}$ is undefined, then we leave $a_{i,j}$ undefined. For each $i \in [1,k]$, let $t_i = t_i(\sigma)$ be the largest index such that a_{i,t_i+1} is defined and define $\mathsf{t}(\sigma) = (t_1, \ldots, t_k)$. As for $\mathsf{s}(\sigma)$, we have that $\mathsf{t}(\sigma) \in \mathcal{WD}_k$ for any $\sigma \in \mathcal{M}_{\Lambda,k}$.

Lemma 12. *For each $\sigma \in \mathcal{M}_{\Lambda,k}$ and each $i \in [1,k]$, we have that $a_{i,t_i+1} \leqslant b_{i,1} = b_i$. In addition, $a_{i,t_i+1} = b_{i,1}$ whenever $t_i < t_{i-1}$ or $i = 1$.*

Remark. As is illustrated in Fig. 8 with $i = 3$, the equality $a_{i,t_i+1} = b_{i,1}$ does not necessarily hold when $t_{i-1} = t_i$.

Proof. The first statement is obvious by definition. For the second statement, if $t < t_{i-1}$ and $a_{i,t+1} < b_{i,1}$, then $a_{i,t+2}$ is defined, which implies that t cannot be equal to t_i. □

For $\mathsf{s} \in \mathcal{WD}_k$, define $\varepsilon(\mathsf{s})$ as the integer with the property that $s_{\varepsilon(\mathsf{s})} > 0$ and $s_{\varepsilon(\mathsf{s})+1} = 0$; $\varepsilon(\mathsf{s}) = 0$ if $s_1 = 0$ and $\varepsilon(\mathsf{s}) = k$ if $s_k \neq 0$. The first $\varepsilon(\mathsf{s})$ coefficients $s_1, \ldots, s_{\varepsilon(\mathsf{s})}$ of s are non-zero, whereas the other coefficients are zero, meaning that $b_{i,1}(\sigma) = n-k+i$. For a face $\sigma \in \mathcal{M}_{\Lambda,k}$, we will write $\varepsilon(\sigma) = \varepsilon(\mathsf{s}(\sigma))$.

For $\Lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_j \geqslant k$ for all j and $\mathsf{s} \in \mathcal{WD}_k$, let $\mathcal{M}_{\Lambda,k}(\mathsf{s})$ be the family of faces $\sigma \in \mathcal{M}_{\Lambda,k}$ with $\mathsf{s}(\sigma) = \mathsf{s}$. For Λ such that $\lambda_j \geqslant k+1$ for all j and $\mathsf{t} \in \mathcal{WD}_k$,

					b_1		b_2				b_3						
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	-	-	-	-	**1**	-	-	-	**1**	-	-	-	**1**	**1**	1	1	1
1	**1**	**1**	**1**	-	1	-	**1**	-	1	-	-	**1**	1	-	**1**	1	1
2	-	**1**	**1**	-	-	-	1	-	-	-	**1**	1	**1**	-	1	-	1
3	-	-	**1**	-	**1**	1	-	-	1	-	1	-	-	-	-	-	-

$\sigma \cap ([0, 3] \times [1, n])$

$b_{i,j}$	$j=1$	2	3	4	5
$i=1$	**5**	**9**	**13**	**14**	15
2	**7**	**12**	**15**	16	-
3	**11**	**13**	17	-	-

$s(\sigma) = (4, 3, 2)$

$a_{i,j}$	$j=1$	2	3	4
$i=1$	**1**	**2**	**3**	5
2	**2**	**3**	7	-
3	**3**	**5**	9	-

$t(\sigma) = (3, 2, 2)$

Fig. 8. A face $\sigma \in \mathcal{M}_{\Lambda,3}$ restricted to $[0, 3] \times [1, n]$ (this example is not authentic). Bold elements in the leftmost region are part of diagonals counted by t; bold elements in the mid-region are part of diagonals counted by s.

let $\mathcal{M}_{\Lambda,k}(\mathsf{s}, \mathsf{t})$ be the subfamily of $\mathcal{M}_{\Lambda,k}(\mathsf{s})$ consisting of all faces σ with $\mathsf{t}(\sigma) = \mathsf{t}$. Define

$$\mathcal{M}_{\Lambda,k}(\mathsf{s}, \geqslant \mathsf{t}) = \bigcup_{\mathsf{t}' \geqslant \mathsf{t}} \mathcal{M}_{\Lambda,k}(\mathsf{s}, \mathsf{t}').$$

Our refined theorem is as follows:

Theorem 13. *Let* $1 \leqslant k \leqslant n$ *and let* $\Lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\Lambda' = (\lambda'_1, \ldots, \lambda'_n)$ *be stack polyominoes with the same content. Then the following hold*:

(i) *If* $\lambda_j \geqslant k$ *for all* $j \in [1, n]$ *and* $\mathsf{s} \in \mathcal{WD}_k$, *then*

$$|\mathcal{M}_{\Lambda,k}(\mathsf{s})| = |\mathcal{M}_{\Lambda',k}(\mathsf{s})|.$$

(ii) *If* $\lambda_j \geqslant k + 1$ *for all* $j \in [1, n]$, $\mathsf{s} \in \mathcal{WD}_k$, $\mathsf{t} \in \mathcal{WD}_k$, *and* $t_1 = \ldots = t_{\varepsilon(\mathsf{s})}$, *then*

$$|\mathcal{M}_{\Lambda,k}(\mathsf{s}, \geqslant \mathsf{t})| = |\mathcal{M}_{\Lambda',k}(\mathsf{s}, \geqslant \mathsf{t})|.$$

We prove Theorem 13 in Section 5. See Section 6 for a further refinement.

Remark. In part (ii), we need the additional requirement on t, because t_i has certain undesirable properties for $i < \varepsilon(\mathsf{s})$. Specifically, we need the requirement in Section 5.3.2 to be able to conclude the proof of the theorem.

4.2. Consequences of Theorem 13

Using Theorem 13, we easily deduce the main result of this paper:

Theorem 14. *Let* $k \geqslant 1$ *and let* Λ *and* Λ' *be stack polyominoes with the same content. Then* $|\mathcal{M}_{\Lambda,k}| = |\mathcal{M}_{\Lambda',k}|$.

Proof. Assume that the column support of Λ is $[1, n]$. If $n \leqslant k+1$, then all elements in $\Delta_{\Lambda,k}$ and $\Delta_{\Lambda',k}$ are cone points; hence the theorem trivially holds in this case. If $\lambda_j \leqslant k$ for some j, then all elements in this column are cone points; hence we may ignore the column. Finally, if $n \geqslant k + 1$ and $\lambda_j \geqslant k + 1$ for all $j \in [1, n]$, then we may apply part (i) of Theorem 13. Summing over all $\mathsf{s} \in \mathcal{WD}_k$, we immediately obtain the desired result. $\square$

Let us consider the special case $\Delta_{n,k}$. Let $C_m = \frac{1}{m+1}\binom{2m}{m}$; C_m is the mth Catalan number. Recall that $A_n = (n-1, \dots, 2, 1)$.

Theorem 15 (*Herzog and Trung [14]*). *For $k \geqslant 1$ and $n \geqslant 2k+1$, $|\mathcal{M}_{A_n,k}|$ is equal to the determinant*

$$\det(C_{n-i-j})_{i,j\in[1,k]} = \begin{vmatrix} C_{n-2} & C_{n-3} & \dots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \dots & C_{n-k-1} & C_{n-k-2} \\ \dots & \dots & \dots & \dots & \dots \\ C_{n-k-1} & C_{n-k-2} & \dots & C_{n-2k+1} & C_{n-2k} \end{vmatrix}.$$

See also [13].

Corollary 16. *For $k \geqslant 1$ and $n \geqslant 2k+1$, $|\mathcal{M}_{n,k}|$ is equal to the determinant* $\det(C_{n-i-j})_{i,j\in[1,k]}$ *in Theorem* 15.

Proof. With $\Lambda = \Omega_n = (1, 2, \dots, n-1)$ and $\Lambda' = A_n = (n-1, \dots, 2, 1)$, Theorem 14 implies that $|\mathcal{M}_{\Omega_n,k}| = |\mathcal{M}_{A_n,k}|$. By Theorem 15, we are done. $\square$

Using the quotient-difference algorithm, Viennot [23] proved that the determinant in Corollary 16 coincides with the expression in the following corollary; see [13, Chapter 4] for a more direct proof.

Corollary 17. *For $k \geqslant 1$ and $n \geqslant 2k+1$,*

$$|\mathcal{M}_{n,k}| = \prod_{1\leqslant i\leqslant j\leqslant n-2k-1} \frac{i+j+2k}{i+j} = |\mathcal{M}_{n-1,k}| \cdot \prod_{r=1}^{2k} \frac{2(n-2k-1)+r}{n-2k-1+r}.$$

5. Proof of Theorem 13

We will prove Theorem 13 by induction on the size of Λ. The base cases are $\Lambda = (k, \dots, k)$ and $n = k$. In both cases, we have nothing but cone points, which implies that the theorem is trivially true. There are three cases for the induction step:

(1) $\lambda_j = k$ for some (but not all) j and $n > k$; see Section 5.1.
(2) $\lambda_j \geqslant k+1$ for all j, $n > k$, and $\mathsf{s} = 0$; see Section 5.2.
(3) $\lambda_j \geqslant k+1$ for all j, $n > k$, and $\mathsf{s} \neq 0$; see Section 5.3.

5.1. *The cases $\lambda_1 = k$ and $\lambda_n = k$*

In this section, we assume that $\lambda_\alpha = k$ for some α, either $\alpha = 1$ or $\alpha = n$. Note that all elements in column α are cone points, so they are all present in each maximal face. For (ii) in the theorem, there is nothing to prove; thus consider (i). Consider the polyomino $\Lambda^{\alpha\leftarrow}$

that we obtain by removing the αth column in Λ and moving everything to the right of column α one step to the left. We want to prove that

$$|\mathcal{M}_{\Lambda,k}(\mathsf{s})| = |\mathcal{M}_{\Lambda^{\alpha\leftarrow},k}(\mathsf{s} - \mathbb{I})|,$$

where $\mathbb{I}$ is the vector of length k with ones on all positions; $\mathbb{I} = (1, 1, \ldots, 1)$. By symmetry, the same identity will hold for Λ'; hence (i) follows by induction. Let $\sigma \in \mathcal{M}_{\Lambda,k}(\mathsf{s})$ and let $\hat{\sigma}$ be the corresponding face in $\mathcal{M}_{\Lambda^{\alpha\leftarrow},k}$ that we obtain by removing column α.

First, suppose that $\alpha = n$. Then the elements $(i-1)(n-1-k+i)$ forming the $[0, k-1]$-diagonal just to the left of the rightmost $[0, k-1]$-diagonal $\{(i-1)(n-k+i) : i \in [1, k]\}$ are clearly all cone points in $\Delta_{\Lambda,k}$. In particular, with $b_{i,j} = b_{i,j}(\sigma)$ defined as in (5), we have that $b_{r,s_r(\sigma)} = n - 1 - k + r$. Namely, we certainly have that $b_{r,s_r} \leqslant n - 1 - k + r$, and if $b_{r,s_r} < n - 1 - k + r$, then we must have that $b_{i,s_r} < n - 1 - k + i$ for all $i \leqslant r$. However, this is a contradiction, as we would then have that $b_{i,s_r+1} \leqslant n - 1 - k + i$ for all $i \leqslant r$, contradicting Lemma 11. Thus $s_r(\hat{\sigma}) = s_r(\sigma) - 1$.

Next, suppose that $\alpha = 1$. To prove that $s_i(\hat{\sigma}) = s_i(\sigma) - 1$, we show that $b_{i,j}(\hat{\sigma}) = b_{i,j+1}(\sigma) - 1$ for $i \in [1, k]$ and $j \in [1, s_i(\sigma)]$; note that everything is shifted one step to the left when we go from σ to $\hat{\sigma}$. It suffices to prove that $b_{i,1}(\hat{\sigma}) = b_{i,2}(\sigma) - 1$ for $i \in [1, k]$. This is clear for $i = 1$. Assume that $i > 1$ and that $b_{i-1,1}(\hat{\sigma}) = b_{i-1,2}(\sigma) - 1$. By Lemma 6 applied to $\hat{\sigma}$ and $p = i - 1$, we have that $(i-1)b_{i-1,1}(\hat{\sigma}) \in \hat{\sigma}$ and hence that $(i-1)b_{i-1,2}(\sigma) \in \sigma$. This implies by construction that $b_{i-1,2}(\sigma) \notin (b_{i-1,1}(\sigma), b_{i,1}(\sigma))$; there are no edges $(i-1)y \in \sigma$ with y in this open interval. In particular, $b_{i-1,2}(\sigma) \geqslant b_{i,1}(\sigma)$, which implies that

$$\begin{aligned} b_{i,2}(\sigma) &= \min\{b : b > b_{i-1,2}(\sigma), (i-1)b \in \sigma\} \\ &= \min\{\hat{b} : \hat{b} > b_{i-1,1}(\hat{\sigma}), (i-1)\hat{b} \in \hat{\sigma}\} + 1 = b_{i,1}(\hat{\sigma}) + 1. \end{aligned}$$

Thus we are done by induction.

5.2. *The case* $\mathsf{s} = 0$

From now on, we assume that $\lambda_j \geqslant k + 1$ for all j. For any $\mathsf{t} \in \mathcal{WD}_k$, we want to show that

$$|\mathcal{M}_{\Lambda,k}(0, \mathsf{t})| = |\mathcal{M}_{\Lambda^0_\uparrow,k}(\mathsf{t})|.$$

By symmetry, the same identity will then hold also for Λ'; hence induction yields the desired result.

Note that $b_i(\sigma) = n - k + i$ for $i \in [1, k]$ whenever $\sigma \in \mathcal{M}_{\Lambda,k}(0)$; $s_1 = 0$ implies that $b_1(\sigma) = n-k+1$. By Lemma 9, $|\mathcal{M}_{\Lambda,k}(0)| = |\mathcal{M}_{\Lambda^0_\uparrow,k}|$. Namely, the first link in the lemma is the link over $\Delta_{\Lambda,k}$ with respect to the set $B_0 \cup B_1 = \{(i-1)(n-k+i), i(n-k+i) : i \in [1, k]\}$. This set is present exactly in those maximal faces σ satisfying $\mathsf{s}(\sigma) = 0$. The second link in the lemma is the link over $\Delta_{\Lambda^0_\uparrow,k}$ with respect to the set B_0, but this is a set of cone points.

In fact, we obtain a bijection from $\mathcal{M}_{\Lambda,k}(0)$ to $\mathcal{M}_{\Lambda\setminus(\{0\}\times\Lambda_0),k}$ by simply removing row 0. Hence we obtain a bijection from $\mathcal{M}_{\Lambda,k}(0)$ to $\mathcal{M}_{\Lambda^0_\uparrow,k}$ by first removing row 0 and then replacing each element xy with $(x-1)y$. For a maximal face $\sigma \in \mathcal{M}_{\Lambda,k}(0)$, let $\hat{\sigma}$ denote the face that we obtain from σ via this procedure. Now, note that $b_{i,j}(\hat{\sigma})$ coincides with $a_{i,j}(\sigma)$ for all $i,\ j$ and that the upper bound $b_i(\sigma) = n-k+i$ for $a_{i,j}(\sigma)$ is exactly the upper bound for $b_{i,j}(\hat{\sigma})$. As a consequence, $\mathsf{s}(\hat{\sigma}) = \mathsf{t}(\sigma)$, which concludes the proof.

5.3. *The case* $\mathsf{s} \neq 0$

Let Λ be a stack polyomino containing the rectangle $[0,k] \times [1,n]$. Let $\mathsf{s}, \mathsf{t} \in \mathcal{W}_k$; $\mathsf{s} \neq 0$ and $t_1 = \ldots = t_{\varepsilon(\mathsf{s})}$. Write $\varepsilon = \varepsilon(\mathsf{s})$. We want to prove (ii) in Theorem 13 for the case $\mathsf{s} \neq 0$. More precisely, we want to show that there is a bijection

$$\varphi_\varepsilon : \mathcal{M}_{\Lambda,k}(\mathsf{s}, \geqslant\mathsf{t}) \to \mathcal{M}_{\Lambda,k}(\mathsf{s} - \mathbb{I}_\varepsilon, \geqslant\mathsf{t} + \mathbb{I}_\varepsilon), \tag{7}$$

where $\mathbb{I}_\varepsilon$ is the vector of length k with ones on the first ε positions and zeros on the remaining $k-\varepsilon$ positions; $\mathbb{I}_\varepsilon = (1, \ldots, 1, 0, \ldots, 0)$. By induction on s, this will complete the proof of Theorem 13; we handled the base case $\mathsf{s} = 0$ in the previous section.

To establish the bijection, we first extend the situation to more general moon polyominoes in Section 5.3.1. We need this extension to prove the weaker claim that there is a bijection

$$\varphi_\varepsilon : \bigcup_{s_\varepsilon \neq 0, s_{\varepsilon+1}=0} \mathcal{M}_{n,k}(\mathsf{s}) \to \bigcup_{t_\varepsilon \neq 0, s_{\varepsilon+1}=0} \mathcal{M}_{n,k}(\mathsf{s}, \mathsf{t}),$$

we adopt the convention that $s_{k+1} = 0$ for $\varepsilon = k$. Given this bijection, it suffices to prove, for each s and t, that a face is mapped to $\mathcal{M}_{\Lambda,k}(\mathsf{s} - \mathbb{I}_\varepsilon, \geqslant\mathsf{t} + \mathbb{I}_\varepsilon)$ if and only if it belongs to $\mathcal{M}_{\Lambda,k}(\mathsf{s}, \geqslant\mathsf{t})$. See Section 5.3.2 for this final step.

5.3.1. *Generalization to moon polyominoes*

As it turns out, it will be convenient to consider a more general construction on moon polyominoes; we will refer to the special case that Λ is a stack polyomino containing the rectangle $[0,k] \times [1,n]$) as "our special case". The reason for generalizing is that we will need to consider not only stack polyominoes but also polyominoes that we obtain from stack polyominoes via the transformation $ij \mapsto (\varepsilon - i)(n+1-j)$; this is a 180° rotation followed by a translation.

Let Λ be a moon polyomino. We restrict our attention to the case that $\Delta_{\Lambda,k}$ is pure; as already mentioned before, we do not know whether this is true for all moon polyominoes. As usual, we assume that Λ has column support $[1,n]$ and that $n \geqslant k+1$. Moreover, we assume that the rectangle $R = [0,\varepsilon] \times [1,n]$ is contained in Λ. However, we make no assumptions about Λ_i for $i \in [\varepsilon+1, k]$.

We assume that there are at least $\varepsilon+1$ columns in R containing some element that is not a cone point in $\Delta_{\Lambda,k}$. This is clearly true in our special case; $\{(i-1)i : i \in [0,k]\}$ is a $(k+1)$-diagonal, which implies that the first $\varepsilon+1$ columns in R contain elements that are not cone points. Let $x = x(\Lambda)$ be minimal and $y = y(\Lambda)$ be maximal such that all elements strictly to the left of column x and strictly to the right of column y in R are cone points; by the assumption, we have that $y - x + 1 \geqslant \varepsilon + 1$. Write $R_{\Lambda,\varepsilon} = [0,\varepsilon] \times [x,y]$. We claim that $x = 1$ and $y = n-k+\varepsilon$ in our special case. Namely, the element $\varepsilon(n-k+\varepsilon+1)$ and all elements above it or to the right of it are part of $\Gamma_{\Lambda,k}$ and hence cone points in $\Delta_{\Lambda,k}$. Since 01 and $\varepsilon(n-k+\varepsilon)$ are both contained in $(k+1)$-diagonals, the claim follows.

Let $\mathcal{P}_{\Lambda,k,\varepsilon}$ be the family of faces $\sigma \in \mathcal{M}_{\Lambda,k}$ such that $\sigma \cap R_{\Lambda,\varepsilon}$ contains at least two disjoint $[0, \varepsilon - 1]$-diagonals but no $[0, \varepsilon]$-diagonal. Let $\mathcal{Q}_{\Lambda,k,\varepsilon}$ be the family of faces $\sigma \in \mathcal{M}_{\Lambda,k}$ such that $\sigma \cap R_{\Lambda,\varepsilon}$ contains at least two disjoint $[1, \varepsilon]$-diagonals but no $[0, \varepsilon]$-diagonal. In our special case, the situation is as follows.

Lemma 18. *Let Λ be a stack polyomino containing the rectangle $[0, k] \times [1, n]$. Then*

$$\mathcal{P}_{\Lambda,k,\varepsilon} = \bigcup_{s_\varepsilon \neq 0, s_{\varepsilon+1}=0} \mathcal{M}_{n,k}(\mathsf{s}),$$

$$\mathcal{Q}_{\Lambda,k,\varepsilon} = \bigcup_{t_\varepsilon \neq 0, s_{\varepsilon+1}=0} \mathcal{M}_{n,k}(\mathsf{s}, \mathsf{t}),$$

we adopt the convention that $s_{k+1} = 0$ for $\varepsilon = k$.

Proof. First, note that a $[0, \varepsilon]$-diagonal is contained in $\Lambda \setminus \Gamma_{\Lambda,k}$ if and only if it is contained in $R_{\Lambda,\varepsilon}$, which immediately implies that $s_{\varepsilon+1} = 0$ exactly when there is no $[0, \varepsilon]$-diagonal in $R_{\Lambda,\varepsilon}$.

Next, note that the inequality $s_\varepsilon(\sigma) > 0$ is equivalent to $\sigma \setminus \Gamma_{\Lambda,k}$ containing a $[0, \varepsilon - 1]$-diagonal E. This diagonal is contained in the rectangle $R_{\Lambda,\varepsilon}$, as is the diagonal $E_0 = \{(i-1)(n-k+i) : i \in [1, \varepsilon]\}$. Since $E_0 \subset \Gamma_{\Lambda,k}$, the two diagonals E and E_0 are disjoint as desired. Conversely, given two disjoint $[0, \varepsilon - 1]$-diagonals E' and E' in $R_{\Lambda,\varepsilon}$, we may form two new diagonals with the same union $E \cup E'$; take the smaller element in each row in $E \cup E'$ to form the first one and the larger element from each row to form the second one. The smaller diagonal is clearly strictly to the left of E_0 and hence counted by s_ε.

Finally, $t_\varepsilon > 0$ if and only if there is at least one $[1, \varepsilon]$-diagonal E to the left of the diagonal $E_1 = \{ib_i : i \in [1, \varepsilon]\}$. Since $b_i \leqslant n - k + i$, we have that $E_1 \subset \Lambda \setminus \Gamma_{\Lambda,k}$, which implies that E and E_1 are both diagonals within the rectangle $R_{\Lambda,\varepsilon}$. Conversely, given two disjoint diagonals E and E', we may form a smaller and a larger diagonal as described at the end of the previous paragraph. Then the smaller diagonal is strictly to the left of the diagonal E_1 and hence counted by t_ε. Namely, $s_{\varepsilon+1} = 0$ implies that E_1 is the rightmost $[1, \varepsilon]$-diagonal in $R_{\Lambda,\varepsilon}$; if we had a $[1, \varepsilon]$-diagonal D in $R_{\Lambda,\varepsilon}$ containing rj for some $r \in [1, \varepsilon]$ and $j > b_r$, then we could form a $[0, \varepsilon]$-diagonal from D by replacing the elements on rows 1 through $r - 1$ with the elements ib_{i+1}, $i \in [0, r - 1]$. $\square$

Return to the general case with Λ a moon polyomino with properties as above. We want to show that $\mathcal{P}_{\Lambda,k,\varepsilon}$ and $\mathcal{Q}_{\Lambda,k,\varepsilon}$ have the same size. To achieve this, we define a transformation φ_ε; the case $k = 2$ is illustrated in Fig. 9. Recall the definitions of $x = x(\Lambda)$ and $y = y(\Lambda)$ above. For $\sigma \in \mathcal{P}_{\Lambda,k,\varepsilon}$, define $b_{i,j}$ for $i \in [1, \varepsilon]$ and $j \in \{1, 2\}$ as in (5), except that we use the initial value $b_{1,0} = x - 1$; this makes no difference for our special case. This gives a well-defined value $b_{i,j} \in [x, y]$ for each relevant i, j; there are at least two $[0, \varepsilon - 1]$-diagonals in $\sigma \cap R_{\Lambda,\varepsilon}$ by assumption. Let $b_i(\sigma) = b_{i,1}$ and $c_i(\sigma) = b_{i,2}$. Let

$$\begin{cases} B = B(\sigma) = \{0b_1\} \cup \{ib_{i+1} : i \in [1, \varepsilon - 1], b_{i+1} > c_i\} \subset \sigma, \\ C = C(\sigma) = \{\varepsilon c_\varepsilon\} \cup \{ic_i : i \in [1, \varepsilon - 1], b_{i+1} > c_i\}. \end{cases} \tag{8}$$

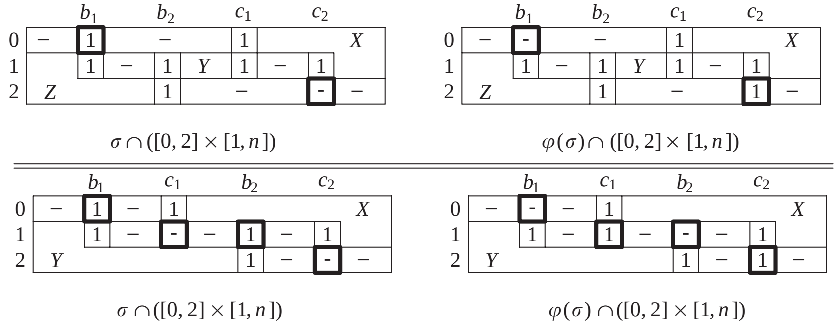

Fig. 9. The two possible cases $b_2 \leqslant c_1$ and $c_1 < b_2$ for $k = \varepsilon = 2$; In each case, squares corresponding to elements in the sets B and C are marked with bold. By Lemma 20, $1c_1 \in \sigma$ in the first case but not in the second case.

Define

$$\varphi_\varepsilon(\sigma) = (\sigma \setminus B(\sigma)) \cup C(\sigma),$$

we replace ib_{i+1} with ic_i whenever $b_{i+1} > c_i$ and replace $0b_1$ with $\varepsilon c_\varepsilon$.

To prove that φ_ε defines a bijection from $\mathcal{P}_{\Lambda,k,\varepsilon}$ to $\mathcal{Q}_{\Lambda,k,\varepsilon}$, we proceed in the following manner:

(1) In Lemma 19, we show that $\varphi_\varepsilon(\sigma)$ is a maximal face in $\Delta_{\Lambda,k}$ whenever $\sigma \in \mathcal{P}_{\Lambda,k,\varepsilon}$.
(2) In Lemmas 20–22, we establish that φ_ε is an injective function from $\mathcal{P}_{\Lambda,k,\varepsilon}$ to $\mathcal{Q}_{\Lambda,k,\varepsilon}$.
(3) We conclude the section with Lemma 23, establishing not only injectivity but also bijectivity. We obtain this via a "dual" argument applied to a certain 180-degree rotation of Λ.

Lemma 19. *For $\sigma \in \mathcal{P}_{\Lambda,k,\varepsilon}$, $\varphi_\varepsilon(\sigma) \in \mathcal{M}_{\Lambda,k}$.*

Proof. First, note that $\sigma \cap C = \emptyset$. Namely, $ic_i \notin \sigma$ if $c_i < b_{i+1}$ for $i < \varepsilon$; σ contains no elements from $\{i\} \times (b_i, b_{i+1})$. Moreover, $\varepsilon c_\varepsilon \notin \sigma$, as otherwise $\{ib_{i+1} : i \in [0, \varepsilon-1]\} \cup \{\varepsilon c_\varepsilon\}$ would be a $[0,\varepsilon]$-diagonal in $R_{\Lambda,\varepsilon}$. In particular, $|\varphi_\varepsilon(\sigma)| = |\sigma|$; we remove as many elements as we add. Since all faces in $\mathcal{M}_{\Lambda,k}$ have the same size by assumption—this is where we need purity—it suffices to prove that $\varphi_\varepsilon(\sigma) \in \Delta_{\Lambda,k}$.

Suppose that $\varphi_\varepsilon(\sigma) \notin \Delta_{\Lambda,k}$. Let E be a $(k+1)$-diagonal in $\varphi_\varepsilon(\sigma)$ such that $C \cap E$ is as small as possible, and let i be such that $ic_i \in C \cap E$. Such an i exists, because σ contains no $(k+1)$-diagonal. For a row index v, let $v \notin E$ mean that no element xy in E satisfies $x = v$. Let

$$\alpha = \max\left(\{0\} \cup \{a : a-1 \notin E, a \in [1,i]\}\right),$$

$\alpha = 0$ or no element from row $\alpha - 1$ is contained in E. For $j \in [\alpha, i]$, let r_j be such that $jr_j \in E$; $r_i = c_i$. Note that $r_j \leqslant c_j$. Namely, by induction on j, $r_j < r_{j+1} \leqslant c_{j+1}$, which implies that $r_j \leqslant \max\{c_j, b_{j+1}\}$. Now, if $b_{j+1} > c_j$, then $jb_{j+1} \notin \varphi_\varepsilon(\sigma)$ ($jb_{j+1} \in B$) and hence $r_j < b_{j+1}$. By definition, there are no elements in σ from $\{j\} \times (b_j, b_{j+1})$, and we deduce that $r_j \leqslant \max\{b_j, c_j\} = c_j$.

Note that if $\alpha = 0$, then $r_0 < r_1 \leqslant c_1$, which gives a contradiction; the only $z < c_1$ such that $0z \in \sigma$ is $z = b_1$, and $0b_1 \in B$. Thus $\alpha > 0$, which implies that no element in E contains $\alpha - 1$. Note that

$$r_\alpha \leqslant c_\alpha < c_{\alpha+1} < \ldots < c_{i-1} < c_i = r_i.$$

Thus we may replace $D = \{jr_j : j \in [\alpha, i]\}$ with $D' = \{(j-1)c_j : j \in [\alpha, i]\}$ and obtain a new $(k+1)$-diagonal $E' = (E \setminus D) \cup D'$ in $\varphi_\varepsilon(\sigma)$; use Lemma 5. However, $C \cap E'$ is strictly smaller than $C \cap E$, because we have removed ic_i without adding any other element from C. This contradicts the minimality of $C \cap E$, and we are done. □

Lemma 20. *We have that $ib_i \in \sigma$ for each $i \in [1, \varepsilon]$. Moreover, with $\tau = \varphi_\varepsilon(\sigma)$, we have that $b_i(\tau) = c_i(\sigma)$ for $i \in [1, \varepsilon]$. In particular, $ic_i(\sigma) \in \sigma$ if and only if $ic_i(\sigma) \notin C$ (i.e., $c_i(\sigma) \geqslant b_{i+1}(\sigma)$) for $i \in [1, \varepsilon]$. As a consequence, $ic_i(\sigma) \in \tau$ for all $i \in [1, \varepsilon]$.*

Proof. The first statement is an immediate consequence of Lemma 6. For the remainder of the lemma, we claim with $c_0(\sigma) = 1$ that

$$c_i = c_i(\sigma) = \min\{c : (i-1)c \in \tau, c > c_{i-1}\}, i \in [1, \varepsilon]. \tag{9}$$

For c_1, this is clear by definition; $0b_1 = 0b_1(\sigma) \in B$ and hence $0b_1 \notin \tau$. For $i > 1$, c_i is minimal such that $(i-1)c_i \in \sigma$ and $c_i > \max\{c_{i-1}, b_i\}$. If $c_{i-1} \geqslant b_i$, then the claim follows immediately. If $c_{i-1} < b_i$, then $(i-1)b_i \in B$ and hence $(i-1)b_i \notin \tau$. Since there is no element in σ from $\{i-1\} \times (b_{i-1}, b_i)$ or from $\{i-1\} \times (b_i, c_i)$ and since $b_{i-1} < c_{i-1}$, it follows that τ does not contain any element from $\{i-1\} \times (c_{i-1}, c_i)$.

As a consequence, $b_i(\tau) = c_i(\sigma)$ as desired. For the remaining statements, note that Lemma 6 implies that we can add $ib_i(\tau) = ic_i(\sigma)$ to τ without introducing any $(k+1)$-diagonals. Since τ is a maximal face, we must hence have that $ic_i(\sigma) \in \tau$. □

Lemma 21. *We have that*

$$\begin{cases} b_\varepsilon(\sigma) = \max\{b : \varepsilon b \in \tau, b < c_\varepsilon(\sigma)\}, \\ b_i(\sigma) = \max\{b : ib \in \tau, b < \min\{c_i(\sigma), b_{i+1}(\sigma)\}\}, \ i \in [1, \varepsilon - 1]. \end{cases} \tag{10}$$

Proof. $\varepsilon c_\varepsilon$ is the only element $\varepsilon j \in \tau \cap R_{\Lambda,\varepsilon}$ such that $j > b_\varepsilon$. For $i < \varepsilon$, assume that the claim is true for $i+1$. Since $b_{i+1} = \min\{b : ib \in \sigma, b > b_i\}$, there is no element in τ from $\{i\} \times (b_i, b_{i+1})$ except possibly ic_i, and we are done. □

Since we can recover $b_i(\sigma)$ and $c_i(\sigma)$ from $\varphi_\varepsilon(\sigma)$ and ε, it is clear that we can reconstruct σ from $\varphi_\varepsilon(\sigma)$ and hence that φ_ε is reversible.

Lemma 22. *φ_ε defines an injection $\mathcal{P}_{\Lambda,k,\varepsilon} \to \mathcal{Q}_{\Lambda,k,\varepsilon}$.*

Proof. We need only prove that φ_ε is a well-defined function between the two families in the lemma; since φ_ε is reversible, this will imply that φ_ε is injective. Let $\sigma \in \mathcal{P}_{\Lambda,k,\varepsilon}$. Then $\varphi_\varepsilon(\sigma)$ contains the two $[1, \varepsilon]$-diagonals $\{ib_i : i \in [1, \varepsilon]\}$ and $\{ic_i : i \in [1, \varepsilon]\}$, and these diagonals are both contained in $R_{\Lambda,\varepsilon}$. Moreover, there is no $[0,\varepsilon]$-diagonal in $\varphi_\varepsilon(\sigma)\cap R_{\Lambda,\varepsilon}$. Namely, by (9), the leftmost $[0,\varepsilon-1]$-diagonal in $R_{\Lambda,\varepsilon}$ is $\{(i-1)c_i : i\in[1,\varepsilon]\}$. If this could be extended to a $[0,\varepsilon]$-diagonal with some element εj, then $\{(i-1)b_i : i\in[1,\varepsilon]\}\cup\{\varepsilon j\}$ would be a $[0,\varepsilon]$-diagonal in $\sigma\cap R_{\Lambda,\varepsilon}$, a contradiction. □

Lemma 23. *φ_ε defines a bijection $\mathcal{P}_{\Lambda,k,\varepsilon}\to\mathcal{Q}_{\Lambda,k,\varepsilon}$.*

Proof. Let Λ^* be the polyomino obtained from Λ via the transformation $ij\to(\varepsilon-i)(n+1-j)$. Since we just rotate the polyomino 180°, it is clear that $\sigma^*\in\mathcal{M}_{\Lambda^*,k}$ if and only if $\sigma\in\mathcal{M}_{\Lambda,k}$, where σ^* is defined in the obvious manner. Also, $[a,b]$-diagonals in $R_{\Lambda,\varepsilon}$ are mapped to $[\varepsilon-b,\varepsilon-a]$-diagonals in $R_{\Lambda^*,\varepsilon}$; $y(\Lambda^*)=n+1-x(\Lambda)$ and $x(\Lambda^*)=n+1-y(\Lambda)$. As a consequence, we have bijections $\mathcal{Q}_{\Lambda,k,\varepsilon}\to\mathcal{P}_{\Lambda^*,k,\varepsilon}$ and $\mathcal{P}_{\Lambda,k,\varepsilon}\to\mathcal{Q}_{\Lambda^*,k,\varepsilon}$. This, together with Lemma 22 applied to each of Λ and Λ^*, implies that

$$|\mathcal{P}_{\Lambda,k,\varepsilon}|\leqslant|\mathcal{Q}_{\Lambda,k,\varepsilon}|=|\mathcal{P}_{\Lambda^*,k,\varepsilon}|\leqslant|\mathcal{Q}_{\Lambda^*,k,\varepsilon}|=|\mathcal{P}_{\Lambda,k,\varepsilon}|.$$

As a consequence, $|\mathcal{P}_{\Lambda,k,\varepsilon}|=|\mathcal{Q}_{\Lambda,k,\varepsilon}|$. The conclusion is that φ_ε must define a bijection between the two sets, and we are done. □

5.3.2. Concluding the proof

The following lemma concludes the proof of Theorem 13.

Lemma 24. *Let Λ be a stack polyomino containing the rectangle $[0,k]\times[1,n]$. Let $\mathsf{s},\mathsf{t}\in\mathcal{W}_k$; $\mathsf{s}\neq 0$ and $t_1=\cdots=t_{\varepsilon(\mathsf{s})}$. Write $\varepsilon=\varepsilon(\mathsf{s})$. Then we can define a bijection*

$$\varphi_\varepsilon:\mathcal{M}_{\Lambda,k}(\mathsf{s},\geqslant\mathsf{t})\to\mathcal{M}_{\Lambda,k}(\mathsf{s}-\mathbb{I}_\varepsilon,\geqslant\mathsf{t}+\mathbb{I}_\varepsilon),$$

where $\mathbb{I}_\varepsilon$ is the vector of length k with ones on the first ε positions and zeros on the remaining $k-\varepsilon$ positions; $\mathbb{I}_\varepsilon=(1,\ldots,1,0,\ldots,0)$.

Proof. For a given maximal face σ, write $\tau=\varphi_\varepsilon(\sigma)$ and $\varepsilon=\varepsilon(\sigma)$. By Lemma 18 and Lemma 23, it suffices to prove the following:

- $\mathsf{s}(\tau)=\mathsf{s}(\sigma)-\mathbb{I}_\varepsilon$.
- $\mathsf{t}(\sigma)\geqslant\mathsf{t}'$ if and only if $\mathsf{t}(\tau)\geqslant\mathsf{t}'+\mathbb{I}_\varepsilon$ whenever $\mathsf{t}'\in\mathcal{WD}_k$ and $t'_1=\cdots=t'_\varepsilon$.

First, consider $\mathsf{s}(\sigma)$. Let $b_{i,j}(\sigma)$ be defined as in (5). We already know that $s_{\varepsilon+1}(\tau)=0$ by Lemmas 18 and 23; thus consider $i\leqslant\varepsilon$. We claim that $b_{i,j}(\tau)=b_{i,j+1}(\sigma)$ for all $i\in[1,\varepsilon]$ and all relevant j; this will imply that $\mathsf{s}(\tau)=\mathsf{s}(\sigma)-\mathbb{I}_\varepsilon$ as desired. Now, $b_{i,1}(\tau)=c_i(\sigma)=b_{i,2}(\sigma)$ for $i\leqslant\varepsilon$ by Lemma 20. Hence the claim follows immediately, as $B(\sigma)$ and $C(\sigma)$ do not contain any elements to the right of $(i-1)c_i(\sigma)$ in row $i-1$.

Next, consider $\mathsf{t}(\sigma)$. Let $a_{i,j}(\sigma)$ be defined as in (6). It is obvious that $a_{i,j}(\tau)=a_{i,j}(\sigma)$ whenever $a_{i,j}(\sigma)$ is defined; $B(\sigma)$ and $C(\sigma)$ do not contain any elements to the left of or equal to $ib_i(\sigma)$ in row i. Since

$$a_{i,t_i(\sigma)+1}(\tau)=a_{i,t_i(\sigma)+1}(\sigma)\leqslant b_i(\sigma)<c_i(\sigma)$$

for all $i \in [1, \varepsilon]$, it is clear that we can go one step further for each $i \in [1, \varepsilon]$, meaning that $a_{i,t_i(\sigma)+2}(\tau)$ can be defined. In particular, $\mathtt{t}(\tau) \geqslant \mathtt{t}(\sigma) + \mathbb{I}_\varepsilon$. However, we do not necessarily have equality. What we do have is the following:

(A) $t_\varepsilon(\tau) = t_\varepsilon(\sigma) + 1$.
(B) $t_r(\tau) = t_r(\sigma)$ whenever $t_r(\sigma) < t_\varepsilon(\sigma)$ and $r > \varepsilon$.

Before proving these claims, we show that they imply that $\mathtt{t}(\tau) \geqslant \mathtt{t}' + \mathbb{I}_\varepsilon$ if and only if $\mathtt{t}(\sigma) \geqslant \mathtt{t}'$ whenever $\mathtt{t}' \in \mathcal{W}_k$ and $t'_1 = \ldots = t'_\varepsilon$. The "if" direction is obvious. For the "only if" direction, we cannot have $t_\varepsilon(\tau) \geqslant t'_\varepsilon + 1$ unless $t_\varepsilon(\sigma) \geqslant t'_\varepsilon$ by (A). Moreover, if $t_\varepsilon(\sigma) \geqslant t'_\varepsilon$ and $t_r(\sigma) < t'_r$, then $t_r(\tau) < t'_r$ by (B), as $t_r(\sigma) < t_\varepsilon(\sigma)$.

Proof of (A). Let $v \in [1, \varepsilon]$ be minimal such that $t = t_v(\sigma) = t_\varepsilon(\sigma)$. By Lemma 12, it is clear that $a_{v,t+1}(\tau) = b_v(\sigma)$ and hence that $a_{v,t+2}(\tau) = \min\{b_{v+1}(\sigma), c_v(\sigma)\}$. Let $\pi \in [v, \varepsilon]$ be minimal such that $c_\pi(\sigma) \leqslant b_{\pi+1}(\sigma)$; this is well-defined, as the inequality holds for $\pi = \varepsilon$. By induction, we obtain that $a_{r,t+2}(\tau) = b_{r+1}(\sigma)$ for $r \in [v, \pi - 1]$. Namely,

$$\begin{aligned} a_{r,t+2}(\tau) &= \min\{a : a > \max\{a_{r,t+1}(\tau), a_{r-1,t+2}(\tau)\}, ra \in \tau\} \\ &= \min\{a : a > b_r(\sigma), ra \in \tau\} = b_{r+1}(\sigma). \end{aligned}$$

The second equality follows from the fact that $a_{r,t+1}(\tau) \leqslant b_r(\sigma) = a_{r-1,t+2}(\tau)$, whereas the last equality follows from the inequality $b_{r+1}(\sigma) \leqslant c_r(\sigma)$. As a consequence,

$$\begin{aligned} a_{\pi,t+2}(\tau) &= \min\{a : a > \max\{a_{\pi,t+1}(\tau), a_{\pi-1,t+2}(\tau)\}, \pi a \in \tau\} \\ &= \min\{a : a > b_\pi(\sigma), \pi a \in \tau\} = c_\pi(\sigma). \end{aligned}$$

By Lemma 20, $c_\pi(\sigma) = b_\pi(\tau)$. Hence $a_{\pi,t+2}(\tau) = b_\pi(\tau)$, which implies that $t_\pi(\tau) = t + 1$. As a consequence, $t_\varepsilon(\tau) = t + 1$ as well, and we are done. □

Proof of (B). By Lemma 12, whenever $t_r(\sigma) < t_{r-1}(\sigma)$ and $r > \varepsilon$, we have that

$$a_{r,t_r(\sigma)+1}(\tau) = a_{r,t_r(\sigma)+1}(\sigma) = b_r(\sigma) = b_r(\tau) = n - k + r,$$

which implies that $t_r(\tau) = t_r(\sigma)$. If $t_r(\sigma) = t_{r-1}(\sigma) < t_\varepsilon(\sigma)$ and $r > \varepsilon$, then

$$t_r(\tau) \geqslant t_r(\sigma) = t_{r-1}(\sigma) = t_{r-1}(\tau) \geqslant t_r(\tau)$$

by induction on r, and we are done. □

6. A further refinement

In this section, we refine Theorem 13 a bit further. Let $k \geqslant 1$ and let Λ be a stack polyomino with row support $[0, p-1]$. For a given face $\sigma \in \mathcal{M}_{\Lambda,k}$ and for $x \in [0, p-1]$, let $\delta_x(\sigma)$ be the number of elements from row x in σ;

$$\delta_x(\sigma) = |\{y : xy \in \sigma\}|.$$

We refer to $\delta(\sigma) = (\delta_0(\sigma), \delta_1(\sigma), \ldots, \delta_{p-1}(\sigma))$ as the *row-degree sequence* of σ. Let $\mathcal{M}^\delta_{\Lambda,k}$ be the subset of $\mathcal{M}_{\Lambda,k}$ consisting of all faces σ with $\delta(\sigma) = \delta$.

With notation as in Theorem 13, for a given sequence $\delta = (\delta_0, \ldots, \delta_{p-1})$, let $\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s})$ be the subset of $\mathcal{M}_{\Lambda,k}(\mathsf{s})$ consisting of all faces with row-degree sequence δ. Define $\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s},\mathsf{t})$ in the analogous manner. It will be convenient to interpret δ as a polynomial; $\delta(x) = \sum_{i=0}^{p-1} \delta_i x^i$.

Theorem 25. *Let $1 \leqslant k \leqslant n$ and let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ and $\Lambda' = (\lambda'_1, \ldots, \lambda'_n)$ be stack polyominoes with the same content. Let $\delta = (\delta_0, \ldots \delta_{p-1})$ be any sequence of positive integers. Then the following hold*:

(i) *If $\lambda_j \geqslant k$ for all $j \in [1, n]$ and $\mathsf{s} \in \mathcal{WD}_k$, then*

$$|\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s})| = |\mathcal{M}^{\delta}_{\Lambda',k}(\mathsf{s})|.$$

(ii) *If $\lambda_j \geqslant k+1$ for all $j \in [1, n]$, $\mathsf{s} \in \mathcal{WD}_k$, and $\mathsf{t} \in \mathcal{WD}_k$ such that $t_1 = \cdots = t_{\varepsilon(\mathsf{s})}$, then*

$$|\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s}, \geqslant \mathsf{t})| = |\mathcal{M}^{\delta}_{\Lambda',k}(\mathsf{s}, \geqslant \mathsf{t})|.$$

Proof. We use the same induction on Λ as in the proof of Theorem 13; see Section 5. The base cases $\Lambda = (k, \ldots, k)$ and $n = k$ are trivial, as all elements are cone points. The other cases are as follows:

(1) $\lambda_j = k$ for some (but not all) j and $n > k$. In the proof in Section 5.1, we remove k cone points in the induction step, which implies that

$$|\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s})| = |\mathcal{M}^{\hat{\delta}}_{\Lambda^{j\leftarrow},k}(\mathsf{s} - \mathbb{1})|,$$

where $\hat{\delta}(x) = \delta(x) - \sum_{i=0}^{k-1} x^i$, and similarly for Λ'.

(2) $\lambda_j \geqslant k + 1$ for all j, $n > k$, and $\mathsf{s} = 0$. In the proof in Section 5.2, we remove k cone points from row 0 in the induction step, which implies that

$$|\mathcal{M}^{\delta}_{\Lambda,k}(0, \mathsf{t})| = |\mathcal{M}^{(\delta(x)-k)/x}_{\Lambda^0_{\uparrow},k}(\mathsf{t})|,$$

and similarly for Λ'.

(3) $\lambda_j \geqslant k + 1$ for all j, $n > k$, and $\mathsf{s} \neq 0$. Let $\varepsilon = \varepsilon(s)$. By the properties of the two sets $B(\sigma)$ and $C(\sigma)$ defined in (8), the bijection φ_ε decreases the number of elements in row 0 and increases the number of elements in row k by one. As a consequence,

$$|\mathcal{M}^{\delta}_{\Lambda,k}(\mathsf{s}, \geqslant \mathsf{t})| = |\mathcal{M}^{\delta-1+x^{\varepsilon}}_{\Lambda,k}(\mathsf{s} - \mathbb{1}_\varepsilon, \geqslant \mathsf{t} + \mathbb{1}_\varepsilon)|,$$

and similarly for Λ'. $\square$

Corollary 26. *Let $k \geqslant 1$, let Λ and Λ' be stack polyominoes with the same content and with row support $[0, p-1]$, and let $\delta = (\delta_0, \ldots, \delta_{p-1})$ be any sequence of positive integers. Then $|\mathcal{M}^{\delta}_{\Lambda,k}| = |\mathcal{M}^{\delta}_{\Lambda',k}|$.*

Proof. This is an immediate consequence of Theorem 25; compare to the proof of Theorem 14. $\square$

7. Extensions and counterexamples

Let us consider some natural generalizations of the polyominoes studied in this paper. For each family of polyominoes, we discuss whether Theorem 10 (purity) and Theorem 14 (enumeration) still apply. The *f-polynomial* of a simplicial complex Δ is the polynomial $f(x) = \sum_{i \geqslant 0} f_i x^i$ with the property that f_i is the number of faces in Δ of size i (i.e., dimension $i - 1$).

7.1. Stalactite polyominoes

For the purposes of this section, a *stalactite polyomino* is a column-convex and intersection-free polyomino such that each column is of the form $[0, \lambda - 1]$. We obtain such a polyomino from an ordinary stack polyomino by rearranging the columns in the polyomino in an arbitrary manner. If Λ is a stalactite polyomino and $k = 1$, then Theorems 10 and 14 still hold:

Proposition 27. *Let Λ and Λ' be two stalactite polyominoes with the same content $(\mu_1, \dots, \mu_n)$. Then the f-polynomials of $\Delta_{\Lambda,1}$ and $\Delta_{\Lambda',1}$ coincide. Moreover, both complexes are pure of dimension $\mu_1 + n - 2$.* (Fig. 10)

Proof. For the first claim, it suffices to prove that the *f*-polynomials of $\Delta_{\Lambda,1}$ and $\Delta_{\Lambda',1}$ coincide whenever we can obtain Λ' from Λ via a transposition of two adjacent columns. Let j and $j+1$ be the columns to be swapped and let b be maximal such that $bj, b(j+1) \in \Lambda$. Divide $\Delta_{\Lambda,1}$ into two subfamilies:

- Let $\mathcal{F}_0(\Lambda)$ be the family of faces $\sigma \in \Delta_{\Lambda,1}$ such that σ contains elements from at most one of the two columns in the rectangle $[0, b] \times [j, j+1]$.
- Let $\mathcal{F}_1(\Lambda)$ be the family of faces $\sigma \in \Delta_{\Lambda,1}$ such that σ contains at least one element from each of the two columns in the rectangle $[0, b] \times [j, j+1]$.

We obtain a dimension-preserving bijection from $\mathcal{F}_0(\Lambda)$ to $\mathcal{F}_0(\Lambda')$ by simply swapping the two columns together with their contents. Namely, this operation cannot possibly introduce any 2-diagonals between the two columns, as at least one of the columns in the relevant rectangle $[0, b] \times [j, j+1]$ has empty intersection with σ whenever $\sigma \in \mathcal{F}_0(\Lambda)$.

Let $m = j$ if $\Lambda^{j+1} \subsetneq \Lambda^j$ and $m = j+1$ otherwise; let m' be such that $\{m, m'\} = \{j, j+1\}$. We claim that we obtain a bijection φ from $\mathcal{F}_1(\Lambda)$ to $\mathcal{F}_1(\Lambda')$ by moving the content below row b in column m to column m' and leaving the rectangle $[0, b] \times [j, j+1]$ fixed.

To prove the claim, first note that φ clearly does not introduce any 2-diagonals within the two columns j and $j+1$; the relevant rectangle is kept fixed. Moreover, suppose that $\sigma \in \mathcal{F}_1(\Lambda')$ and that φ does introduce a 2-diagonal. One readily verifies that one of the elements must be of the form $x_1 m'$ for some $x_1 \in [0, b]$, whereas the other element must be of the form $x_2 y$ for some row $x_2 > b$ and some column $y > j+1$. Let $c \geqslant b$ be maximal such that $cm \in \Lambda$ (equivalently, maximal such that $cm' \in \Lambda'$). For $\{x_1 m', x_2 y\}$ to form a 2-diagonal, we must have that $x_2 \in [b+1, c]$. However, by assumption on $\mathcal{F}_1(\Lambda)$, there is an element $x_0 m \in \sigma$ from the interval $[0, b] \times \{m\}$, and $x_0 m$ forms a 2-diagonal together with $x_2 y$ in σ, which is a contradiction.

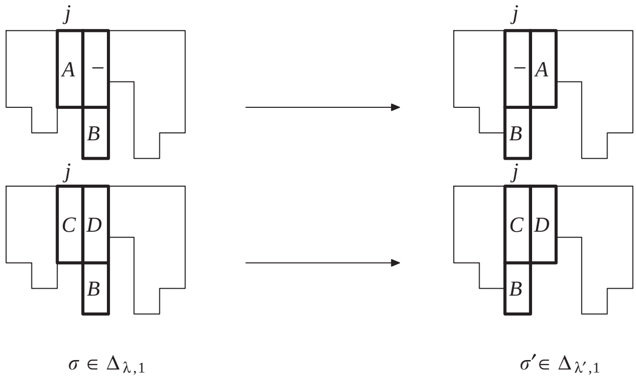

Fig. 10. The two cases in Proposition 27; the upper picture illustrates the bijection $\mathcal{F}_0(\Lambda) \to \mathcal{F}_0(\Lambda')$, whereas the lower picture illustrates the bijection $\mathcal{F}_1(\Lambda) \to \mathcal{F}_1(\Lambda')$. In the latter case, C and D are both non-empty.

	1	2	3	4	5
0	-	1	1	1	1
1	1	1	1	1	1
2	1	1	1	-	-
3	1			1	1

	1	2	3	4	5
0	-	-	-	1	1
1	-	1	1	1	1
2	1	1	1	1	1
3	1			1	1

Fig. 11. Two maximal faces in $\Delta_{(4,3,3,4,4),2}$, one with 15 elements and the other with 14 elements.

For the last statement, note that $k = 1$ is the one case for which the proof of Lemma 9 works without the assumption that $(\lambda_1, \ldots, \lambda_n)$ be unimodal. In particular, we can apply Theorem 10. $\square$

For larger k, the situation is no longer as simple as for $k = 1$. Already for $k = 2$, we have an example showing that $\Delta_{\Lambda,k}$ is not necessarily a pure complex when Λ is a stalactite polyomino. For example, let $\Lambda = (4, 3, 3, 4, 4)$. Then there are maximal faces of two different dimensions; see Fig. 11. In particular, $\Delta_{\Lambda,2}$ is not pure. For those interested, we may mention that the f-polynomial of $\Delta_{\Lambda,2}$ is $(1+8t+28t^2+43t^3+25t^4+t^5)(1+t)^{10}$. The f-polynomial of the related complex $\Delta_{(4,4,4,3,3),2}$ is $(1+8t+28t^2+43t^3+25t^4)(1+t)^{10}$.

Nevertheless, experiments show that other stalactite polyominoes such as (4, 3, 4, 3, 4) and (4, 4, 3, 3, 4) indeed yield pure complexes with the same f-polynomial as $\Delta_{(4,4,4,3,3),2}$. We do not know whether it is possible to generalize this observation to a larger class of stalactite polyominoes.

7.2. Convex polyominoes

In the definition of moon polyominoes, one may consider what would happen if we dropped the condition that the polyomino be intersection-free and only required convexity. Intriguingly, $\Delta_{\Lambda,k}$ is not necessarily pure in this case, not even for $k = 1$. For example, two

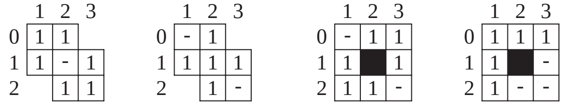

Fig. 12. Two maximal faces in $\Delta_{\Lambda,1}$ for two different polyominoes. In each case, we have one face with six elements and another face with five elements.

maximal faces of different dimensions for a row- and column-convex polyomino are given to the left in Fig. 12.

7.3. *Intersection-free polyominoes*

Finally, consider the situation where we only require that the polyomino be intersection-free. It is not hard to show that if λ is column-convex and intersection-free then the *f*-polynomial of $\Delta_{\lambda,1}$ is invariant under permutations of the rows in λ. Namely, a straightforward generalization of the proof of Proposition 27 is easily seen to go through.

However, the condition about column-convexity cannot be excluded as the rightmost example in Fig. 12 shows; this polyomino λ coincides with (3, 3, 3) with the middle square removed. Again, we have maximal faces of different dimensions in $\Delta_{\lambda,1}$.

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